Chapter two

2.1 The general conduction equation

Consider the one-dimensional system shown in Figure 1-6. If the system is in a steady state, i.e., if the temperature does not change with time, then the problem is a simple one, and we need only integrate Equation (1-1) and substitute the appropriate values to solve for the desired quantity. However, if the temperature of the solid is changing with time, or if there are heat sources or sinks within the solid, the situation is more complex. We consider the general case where the temperature may be changing with time and heat sources may be present within the body. For the element of thickness dx, the following energy balance may be made:

Energy conducted in left face + heat generated within element = change in internal energy + energy conducted out right face These energy quantities are given as follows:

Energy in left face=$q_x = -kA\frac{\partial T}{\partial x}$

Energy generated within element=$\dot{q}Adx$

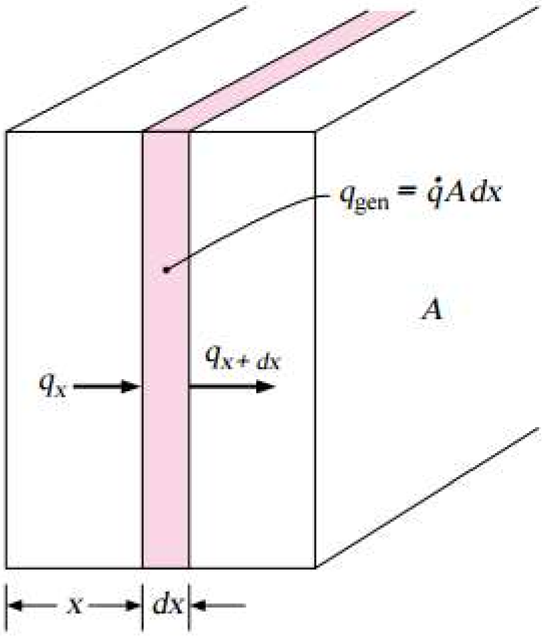


Figure 1-6 Elemental volume for one-dimensional heatconduction analysis.

Change in internal energy=$\rho cA\frac{\partial T}{\partial t}dx$

Energy out right face = $q_{x+dx} = -kA\frac{\partial T}{\partial x}\Big]_{x+dx}$

$$= -A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

where

$\dot{q}$ = energy generated per unit volume, W/m^3
c = specific heat of material, J/kg · °C
ρ = density, kg/m^3

Combining the relations above gives

$$-kA\frac{\partial T}{\partial x} + \dot{q}Adx = \rho cA\frac{\partial T}{\partial t}dx - A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

or

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho c\frac{\partial T}{\partial t} \qquad 1.8$$

This is the one-dimensional heat-conduction equation. To treat more than one-dimensional heat flow, we need consider only the heat conducted in and out of a unit volume in all three coordinate directions, as shown in Figure 1-7a. The energy balance yields

$$q_x + q_y + q_z + q_{gen} = q_{x+dx} + q_{y+dy} + q_{z+dz} + \frac{dE}{dt}$$

And the energy quantities are given by

$$q_x = -kdydz\frac{\partial T}{\partial x}$$

$$q_{x+dx} = -\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]dydz$$

$$q_y = -kdxdz\frac{\partial T}{\partial y}$$

$$q_{y+dy} = -\left[k\frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dy\right]dxdz$$

$$q_z = -kdxdy\frac{\partial T}{\partial z}$$

$$q_{z+dz} = -\left[k\frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)dz\right]dxdy$$

$$q_{gen} = \dot{q}dxdydz$$

$$\frac{dE}{dt} = \rho cdxdydz\frac{\partial T}{\partial t}$$

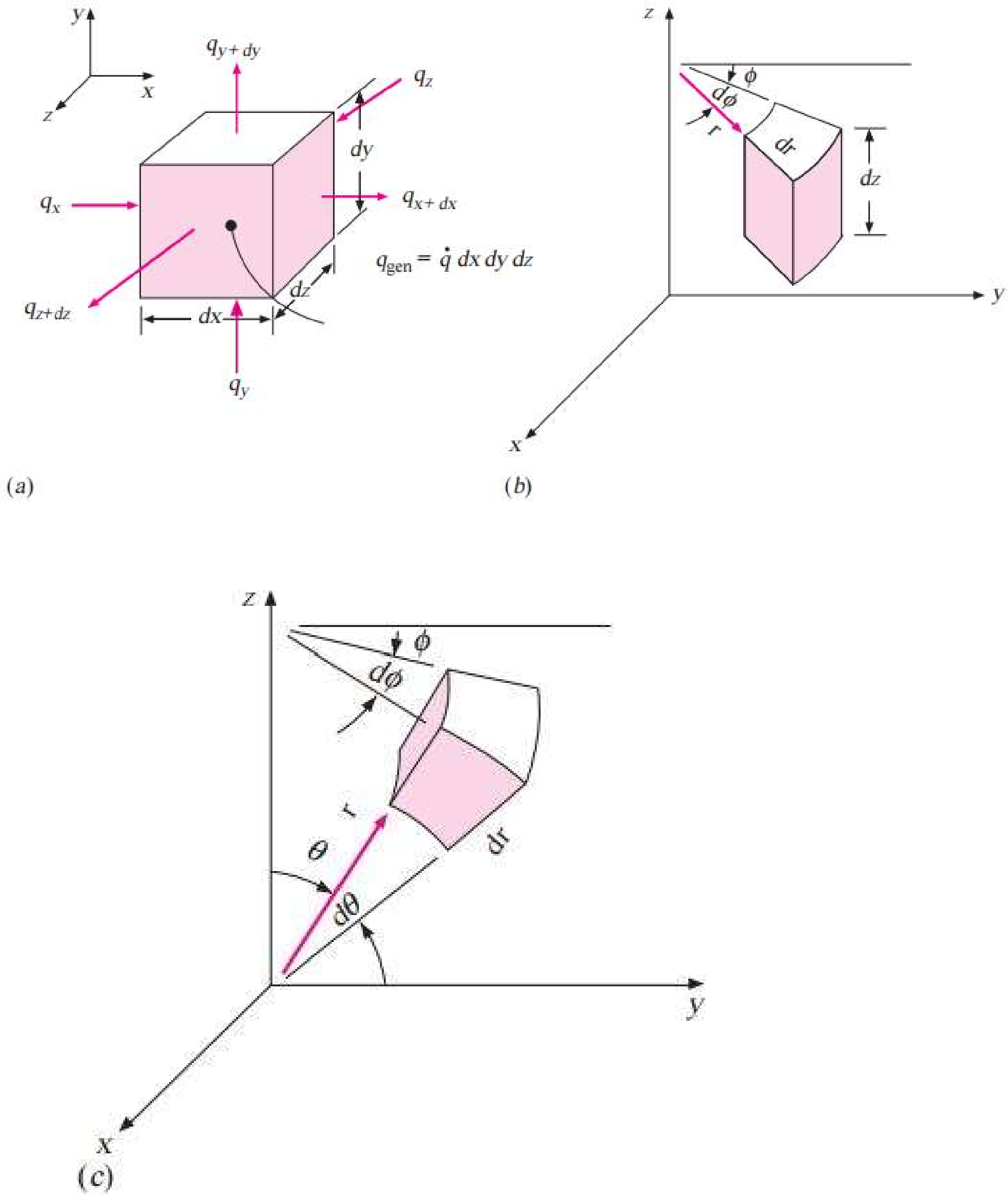


Figure 1.7 Elemental volume for three-dimensional heat-conduction analysis: (a) cartesian coordinates; (b) cylindrical coordinates; (c) spherical coordinates.

So that the general three-dimensional heat-conduction equation is

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t} \qquad 1.9$$

For constant thermal conductivity, Equation (1.9) is written

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad 1.9a$$

Where:

$$\alpha=\frac{\text{Heat conducted}}{\text{Heat stored}}$$

$\alpha=\frac{k}{\rho c}$:is the thermal difusivity of material, m^2/s.

The larger the value of α, the faster heat will diffuse through the material. This may be seen by examining the quantities that make up α. A high value of α could result either from a high value of thermal conductivity, which would indicate a rapid energy-transfer rate, or from a low value of the thermal heat capacity ρc. A low value of the heat capacity would mean that less of the energy moving through the material would be absorbed and used to raise the temperature ofthe material; thus more energy would be available for further transfer. Thermal diffusivity α has units of square meters per second.

Equation (1.9a) may be transformed into either cylindrical or spherical coordinates by standard calculus techniques. The results are as follows:

Cylindrical coordinates: see figure 1.9b

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \emptyset}\left(k\frac{\partial T}{\partial \emptyset}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t} \quad 1.9b$$

For constant thermal conductivity,

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Spherical coordinates: see figure 1.9c

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial \emptyset}\left(k\frac{\partial T}{\partial \emptyset}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\, sin\,\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t} \quad 1.9c$$

For constant thermal conductivity,

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2 T}{\partial \emptyset^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Many practical problems involve only special cases of the general equations listed above. As a guide to the developments in future chapters, it is worthwhile to show the reduced form of the general equations for several cases of practical interest.

Steady-state one-dimensional heat flow (no heat generation):

$$\frac{d^2T}{dx^2} = 0 \quad 1.10$$

Steady-state one-dimensional heat flow in cylindrical coordinates (no heat generation):

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \quad 1.11$$

Steady-state one-dimensional heat flow with heat sources:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \quad 1.12$$

Two-dimensional steady-state conduction without heat sources:

$$\frac{\partial^2T}{\partial x^2} + \frac{\partial^2T}{\partial y^2} = 0 \quad 1.13$$

Example: A 1 m thickness wall made from a material (cp=4000J/kg.k, ρ=1600 kg/m^3, k=40 W/m.k). Temperature distribution is given by $T(x) = a + bx + cx^2$, a=900˚C, b=-300 ˚C/m, c=-50 ˚C/m^2, A=10 m^2, $\dot{q}$=1000 W/m^3.

Determine:

1. Rate of heat transfer entering and leaving the wall.
2. Rate of storing energy within the wall.
3. Rate of change of temperature at x=0.25 m and x=0.5 m.

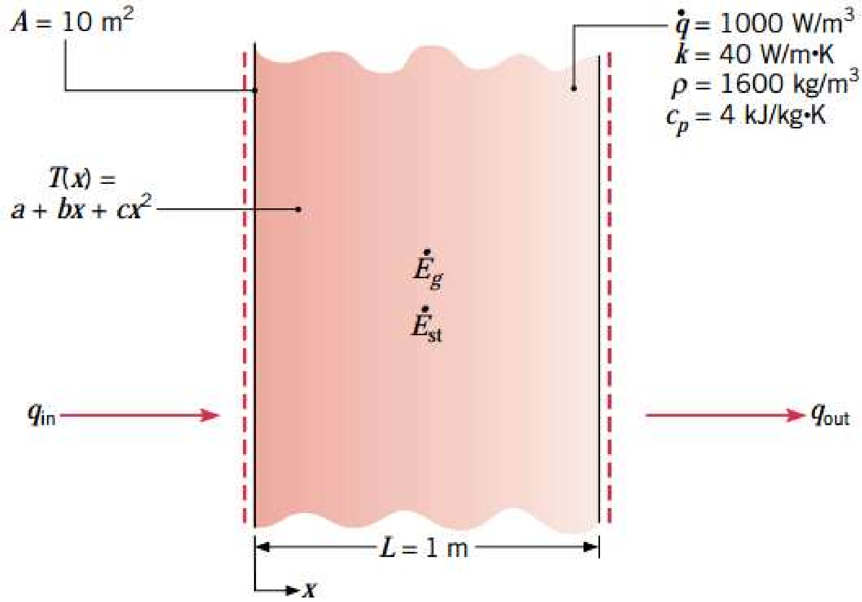
Solution:

Known: Temperature distribution T(x) at an instant of time t in a one-dimensional wall with uniform heat generation.

1. Recall that once the temperature distribution is known for a medium, it is a simple matter to determine the conduction heat transfer rate at any point in the medium, or at its surfaces, by using Fourier's law. Hence the desired heat rates may be determined by using the prescribed temperature distribution with Equation

2.1. Accordingly,

Schematic:



1. Recall that once the temperature distribution is known for a medium, it is a simple matter to determine the conduction heat transfer rate at any point in the medium, or at its surfaces, by using Fourier's law.

$$q_{in} = q_x(0) = -kA\frac{\partial T}{\partial x}\bigg|_{x=0} = -kA(b + 2cx)_{x=0}$$

$$q_{in} = -bkA = 300°\text{C/m} \times 40\ \text{W/m}\cdot\text{K} \times 10\ \text{m}^2 = 120\ \text{kW}$$

Similarly,

$$q_{out} = q_x(L) = -kA\frac{\partial T}{\partial x}\bigg|_{x=L} = -kA(b + 2cx)_{x=L}$$

$$q_{out} = -(b + 2cL)kA = -[-300°\text{C/m}$$

$$+ 2(-50°\text{C/m}^2) \times 1\ \text{m}] \times 40\ \text{W/m}\cdot\text{K} \times 10\ \text{m}^2 = 160\ \text{kW}$$

2.apply energy balance on the wall:

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

where $\dot{E}_g = \dot{q}AL$, it follows that

$$\dot{E}_{st} = \dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = q_{in} + \dot{q}AL - q_{out}$$

$$\dot{E}_{st} = 120\ \text{kW} + 1000\ \text{W/m}^3 \times 10\ \text{m}^2 \times 1\ \text{m} - 160\ \text{kW}$$

medium may be

$$\dot{E}_{st} = -30\ \text{kW}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

From the prescribed temperature distribution, it follows that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(b + 2cx) = 2c = 2(-50°C/m^2) = -100°C/m^2$$

$$\frac{\partial T}{\partial t} = \frac{40\ \text{W/m}\cdot\text{K}}{1600\ \text{kg/m}^3 \times 4\ \text{kJ/kg}\cdot\text{K}} \times (-100°\text{C/m}^2) + \frac{1000\ \text{W/m}^3}{1600\ \text{kg/m}^3 \times 4\ \text{kJ/kg}\cdot\text{K}}$$

$$\frac{\partial T}{\partial t} = -6.25 \times 10^{-4}\text{°C/s} + 1.56 \times 10^{-4}\text{°C/s}$$

$$= -4.69 \times 10^{-4}\text{°C/s}$$

2.2 Study state conduction in one dimension:

2.2.1 The Plane Wall

a. temperature distribution

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

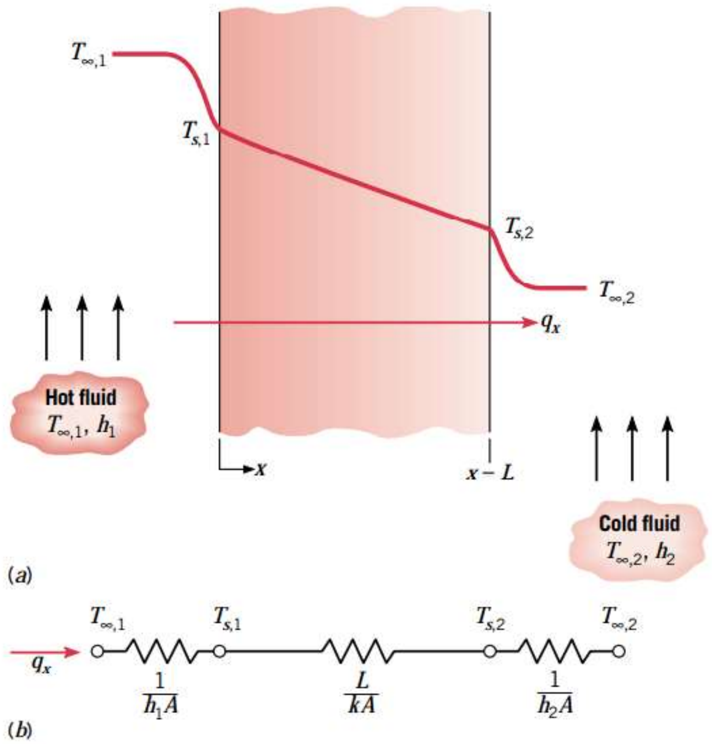


FIGURE 2.1 Heat transfer through a plane wall. (a) Temperature distribution. (b) Equivalent thermal circuit.

Assumption:

Temperature varies with x-direction only no heat generation. So, the general equation is reduces to:

$$\frac{\partial^2 T}{\partial x^2} = 0$$

First integration:

$$\frac{\partial T}{\partial x} = C_1$$

Second integration:

$$T(x) = C_1 x + C_2 \qquad 2.1$$

this is the general solution

C_1 and C_2 are constant of integration.

Boundary condition:

To obtain the constants of integration, C_1 and C_2, boundary conditions must be introduced. We choose to apply conditions of the first kind at $x = 0$ and $x = L$, in which case

$$T(0) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

Applying the condition at $x = 0$ to the general solution, it follows that

$$T_{s,1} = C_2$$

Similarly, at $x = L$,

$$T_{s,2} = C_1L + C_2 = C_1L + T_{s,1}$$

in which case

$$\frac{T_{s,2} - T_{s,1}}{L} = C_1$$

Substitute in general solution:

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1} \qquad 2.2$$

the heat transfer rate:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2}) \qquad 2.3$$

b. Thermal resistance

The thermal resistance for conduction is:

$$R_{t,\,cond} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA} \qquad 2.4$$

The thermal resistance for convection is:

$$R_{t,\,conv} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA} \qquad 2.5$$

Circuit representations provide a useful tool for both conceptualizing and quantifying heat transfer problems. The equivalent thermal circuit for the plane wall with convection surface conditions is shown in Figure 2.1b. The heat transfer rate may be determined from separate consideration of each element in the network. Since qx is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

In terms of the *overall temperature difference*, $T_{\infty,1} - T_{\infty,2}$, and the *total thermal resistance*, R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

Because the conduction and convection resistances are in series and may be summed, it follows that

$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

2.1.2 The Composite Wall:

a. Material in series:

Equivalent thermal circuits may also be used for more complex systems, such as composite walls. Such walls may involve any number of series and parallel thermalresistances due to layers of different materials. Consider the series composite wall of Figure 2.2. The one-dimensional heat transfer rate for this system may be expressed as

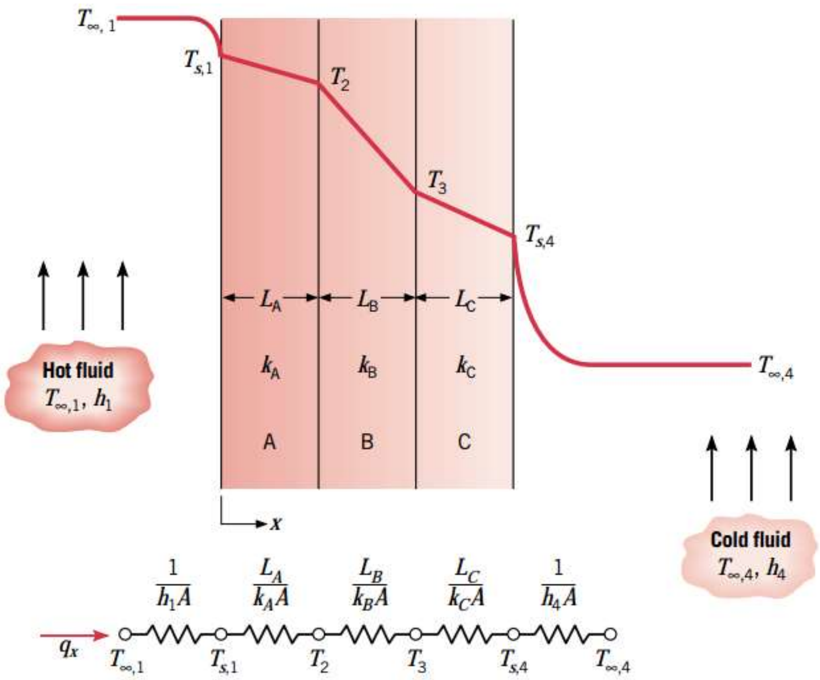


FIGURE 2.2 Equivalent thermal circuit for a series composite wall.